Meeting 3 Math 22

Section 9.2.3 Operations on Vectors

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ and $c \in \mathbb{R}$.

Addition: $u + v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar Mult: $cu = \langle cu_1, cu_2, cu_3 \rangle$

Subtraction: $u - v = u + (-1)v$

Fact: Every vector $v = \langle a, b, c \rangle$ is a sum of scalar multiples of the standard unit vectors

$i = \langle 1,0,0 \rangle \quad j = \langle 0,1,0 \rangle \quad k = \langle 0,0,1 \rangle$.

We have

$$v = \langle a,b,c \rangle = \langle a,0,0 \rangle + \langle 0,b,0 \rangle + \langle 0,0,c \rangle = ai + bj + ck.$$

Section 9.2.4 Properties of Vector Operations

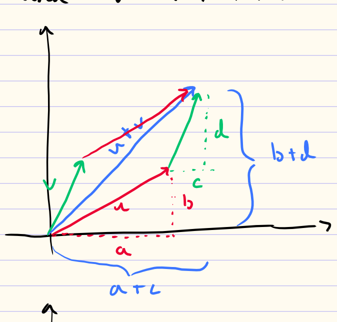
Let u, v, w be vectors in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$. Then

1. $u + v = v + u$
2. $(u+v) + w = u + (v+w)$
3. The zero vector $0 = \langle 0, \dots, 0 \rangle$ satisfies $0 + v = v = v + 0$
4. $v + (-1)v = 0 = (-1)v + v$. We write $-v = (-1)v$ and call $-v$ the additive inverse of v.
5. $(a+b)v = av + bv$
6. $a(u+v) = au + av$
7. $(ab)v = a(bv)$
8. $1 \cdot v = v$

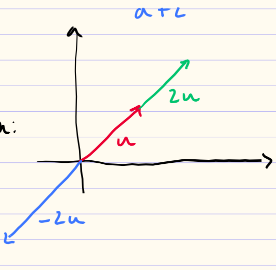
Section 9.2.5 Geometric Interpretation of Vector Operations

Let $u = \langle a,b \rangle$ and $v = \langle c,d \rangle \Rightarrow u+v \langle a+c, b+d \rangle$

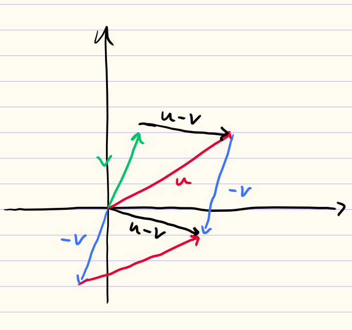
Addition:



Scalar Multiplication:



Subtraction:



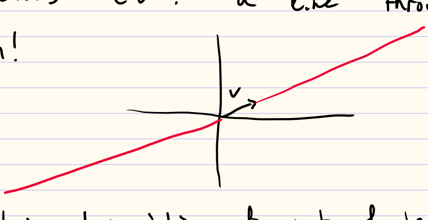
Activity 9.2.4 (20 min)

- Complete Activity 9.2.4 and discuss with your group.
- Class discussion.

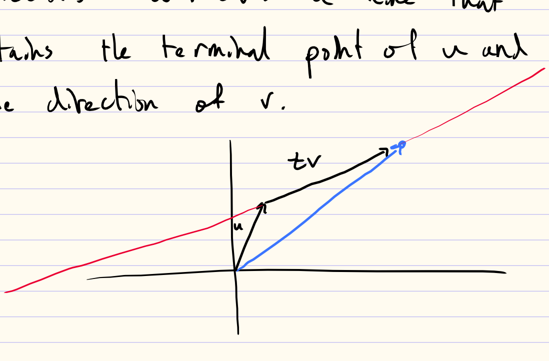
What is $0 \cdot v$? Say $v = \langle a, b \rangle$

$0 \cdot v = \langle 0 \cdot a, 0 \cdot b \rangle = \langle 0, 0 \rangle$

Geometric description of set of terminal points of vectors tv: a line through the origin!



Geometric description of set of terminal points of vectors $u + tv$: a line that contains the terminal point of u and runs in the direction of v.



Section 9.2.6 The Magnitude of a Vector

Activity 9.2.5 (20 min)

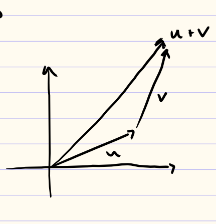
- Complete Activity 9.2.5 and discuss with your group.
- Class discussion.

c. Formula for $|v|$ if $v = \langle v_1, v_2, v_3 \rangle$:

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

d. When is $|u| + |v| = |u+v|$?

If u and v are parallel!



g. Relation between $|tv|$ and $|v|$:

$$|tv| = |t| \cdot |v|$$

i. Unit vector in the direction of v: $\frac{v}{|v|}$.

Proof: $\left| \frac{v}{|v|} \right| = \left| \frac{1}{|v|} \right| \cdot |v| = \frac{|v|}{|v|} = 1.$

End of Section 9.2

Section 9.3 The Dot Product

Reading Debrief (8-10 min)

- Discuss Sections 9.3.1/9.3.2 with your group. Ask any questions you wrote down (see #5 on Reading 2).
- Are there any questions you want me to address today?

Questions:

- $i \cdot i = \langle 1,0 \rangle \cdot \langle 1,0 \rangle = 1$
- $i \cdot j = \langle 1,0 \rangle \cdot \langle 0,1 \rangle = 0$

Section 9.3.1 The Dot Product

Definition 9.3.4 The dot product of $u = \langle u_1, \dots, u_n \rangle$ and $v = \langle v_1, \dots, v_n \rangle$ is the real number

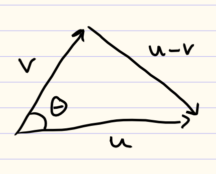
$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Properties of the Dot Product

1. $u \cdot v = v \cdot u$
2. $(u+v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v)$
4. $u \cdot u = |u|^2$

Section 9.3.2 The angle between vectors

Any 2 vectors u, v determine a triangle



The Law of Cosines relates the quantities $|u|, |v|, |u-v|, \theta$. You will get

$$u \cdot v = |u||v| \cos\theta$$

or

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right).$$

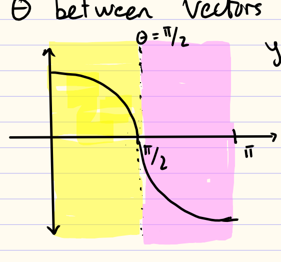
Activity 9.3.3 (20 min)

- Complete Activity 9.3.3 and discuss with your group.
- Class discussion.

Section 9.3.3 The Dot Product & Orthogonality

Summary of Activity 9.3.3

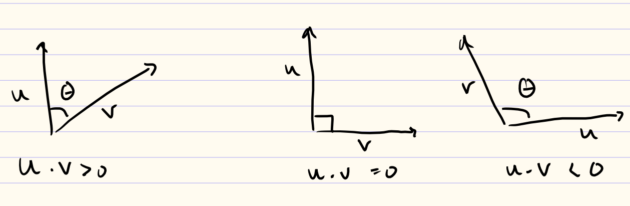
The angle θ between vectors satisfies $0 \le \theta \le \pi$.



$u \cdot v = 0$ if and only if $\cos\theta = 0$ if and only if $\theta = 90^\circ$.

$u \cdot v > 0$ if and only if $\cos\theta > 0$ if and only if θ is acute

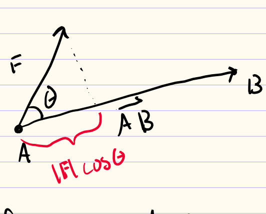
$u \cdot v < 0$ if and only if $\cos\theta < 0$ if and only if θ is obtuse.



Section 9.3.4 Work, Force, Displacement

Work is a measure of the energy to displace an object with a force. According to the physicists, the work W required to displace an object from A to B with a force F is given by

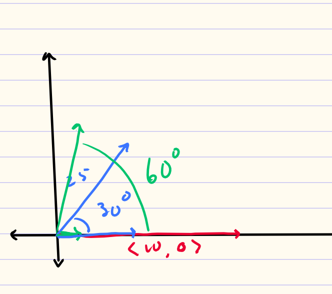
$$W = F \cdot \vec{AB} = |F||\vec{AB}|\cos\theta.$$



According to the formula and by trigonometry, the work W only depends on the magnitude of the force parallel to the displacement.

Activity 9.3.4 (15 min)

- Complete 9.3.4 and discuss w/ your group.
- Class discussion.



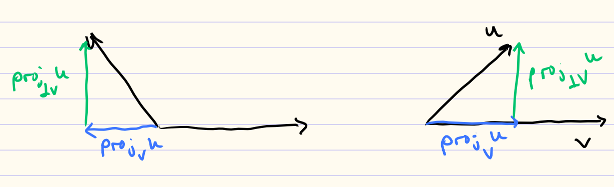
$W = 25 \cdot 10 \cdot \cos 30^\circ = 125\sqrt{3}$

Section 9.3.5 Projections

Motivation: Given vectors u and v, we want to write u as a sum

$$u = \text{proj}_v u + \text{proj}_{\perp v} u$$

where $\text{proj}_v u$ is parallel to v and $\text{proj}_{\perp v} u$ is perpendicular to v.



We have $u = \text{proj}_v u + \text{proj}_{\perp v} u$.

Since $\text{proj}_{\perp v} u$ is perp. to v, we get

$$0 = \text{proj}_{\perp v} u \cdot v = (u - \text{proj}_v u) \cdot v = u \cdot v - \text{proj}_v u \cdot v.$$

Since v is parallel to $\text{proj}_v u$, there is some scalar K such that $\text{proj}_v u = Kv$. Then

$$K = \frac{u \cdot v}{v \cdot v}.$$

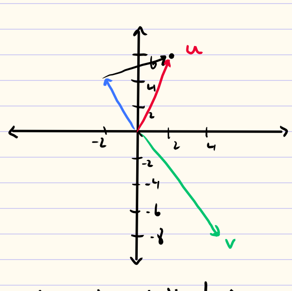
Then $\text{proj}_v u = Kv = \frac{u \cdot v}{v \cdot v} v$. — The Projection of u onto v

We also define $\text{comp}_v u = \frac{u \cdot v}{|v|}$.

Then $\text{proj}_v u = \frac{u \cdot v}{|v|} \cdot \frac{v}{|v|} = \text{comp}_v u \, \frac{v}{|v|}$. — The Component of u along v; $\frac{v}{|v|}$: a unit vector

Activity 9.3.5 (20 min)

- Complete Activity 9.3.5 and discuss with your group.
- Class discussion.



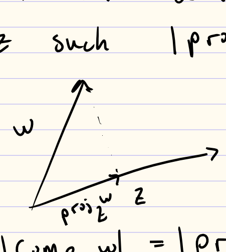
$\text{proj}_v u = \langle -2, 4 \rangle$

$\text{proj}_{\perp v} u = \langle 4, 2 \rangle$

$\text{comp}_v u = -4.47$

b. Given $v = \langle -2, 4 \rangle$. What is $\text{proj}_v u$? Since v is parallel to the green vector so the projection is the same as in a.

c. Given $z = \langle 3, 4 \rangle$. Find a vector w not parallel to z such $|\text{proj}_z w| = 10$.

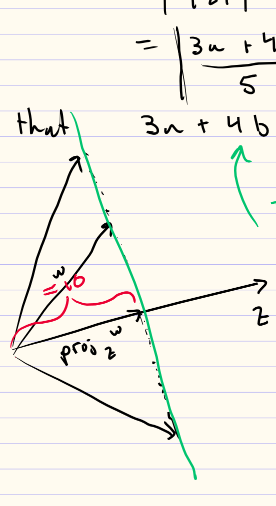


We know that $|\text{comp}_z w| = |\text{proj}_z w|$

Say $w = \langle a, b \rangle$. We want

$$10 = |\text{comp}_z w| = \left| \frac{w \cdot z}{|z|} \right| = \left| \frac{3a + 4b}{5} \right|$$

Find a, b such that $3a + 4b = 50$. — This eq. describes a line perpendicular to z.



End of Section 9.3